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Completely integrable systems: Jacobi's heritage

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Abstract

During the last few decades, algebraic geometry has become a tool for solving differential equations and spectral questions of mechanics and mathematical physics. This paper deals with the study of the integrable systems from the point of view of algebraic geometry, inverse spectral problems and mechanics from the point of view of Lie groups. Section 1 is preliminary giving a little background. In Section 2, we study a Lie algebra theoretical method leading to completely integrable systems, based on the Kostant–Kirillov coadjoint action. Section 3 is devoted to illustrate how to decide about the algebraic complete integrability (a.c.i.) of Hamiltonian systems. Algebraic integrability means that the system is completely integrable in the sense of the phase space being foliated by tori, which in addition are real parts of a complex algebraic tori (abelian varieties). Adler–van Moerbeke's method is a very useful tool not only to discover among families of Hamiltonian systems those which are a.c.i., but also to characterize and describe the algebraic nature of the invariant tori (periods, etc.) for the a.c.i. systems. Some integrable systems, such as Kortewege–de Vries equation, Toda lattice, Euler rigid body motion, Kowalewski's top, Manakov's geodesic flow on $SO(4)$, etc. are treated. © 1999 Elsevier Science B.V. All rights reserved.

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1. Background

A Hamiltonian system $X_H : \dot{z} = \partial H / \partial z$, $J \equiv J(z)$ anti-symmetric, possibly depending on $z \in \mathbb{R}^{2n}$, $\dot{\equiv} d/dt$ is called *completely integrable* if there exist n integrals (or conserved quantities) $H_1 = H, H_2, \dots, H_n$ in involution (i.e. such that the Poisson brackets $\{H_i, H_j\}$ all vanish) with linearly independent gradients (i.e. dH_1, \dots, dH_n linearly independent). For most values of $c_i \in \mathbb{R}$, the invariant manifolds

$$\bigcap_{i=1}^n \{H_i = c_i, z \in \mathbb{R}^{2n}\}$$

are compact, connected and therefore diffeomorphic to real tori $\mathbb{R}^n / \text{lattice}$ by the Arnold–Liouville theorem [7]. Also, there is a transformation to so-called action–angle variables, mapping the flow into a straight line motion on that torus.

Last century, mechanics was dominated by the question whether a dynamical system can be solved by quadratures, i.e. by a finite number of algebraic operations including the inverting of functions. This was done, if at all possible, by finding appropriate variables $(\alpha_1, \dots, \alpha_2)$ such that $(\alpha_1, \dots, \alpha_n, H_1, \dots, H_n)$ form a local system of canonical coordinates in which the Hamiltonian vector fields X_{H_i} take the simple form

$$X_{H_j} : \begin{cases} \dot{\alpha}_i = \delta_{ij}, \\ \dot{H}_i = 0 \end{cases}$$

Historically, the developments of mechanics and algebraic geometry (in particular the theory of Riemann surfaces) were closely intertwined. This comes from the fact that, in most examples, the quadratures α_i were obtained in terms of elliptic or hyperelliptic integrals

$$\alpha_i = \sum_{k=1}^g \int_{s_k(t)}^{s_k(t)} \frac{z^{n-i} dz}{\sqrt{P(z, c_j)}} \quad (1.1)$$

with $P(z, c_j)$ a polynomial of degree $2g + 1$ or $2g + 2$ in z and the $s_k(t)$ some appropriate variables, algebraically related to the originally given ones, for which the Hamilton–Jacobi equation could be solved by separation of variables. The solutions of these problems can be expressed in terms of θ -functions related to Riemann surfaces. Some examples of this are:

(a) Jacobi’s integration [16] of the geodesics on ellipsoid by using elliptic coordinates and various tricks. (b) Neumann’s study [45] of a mass point moving on the sphere under the influence of a linear force, using the spherical elliptic coordinates. (c) Euler, Lagrange and Kowalewski [21] consider the problem of some three-dimensional rigid body motions and they express their solutions in terms of elliptic and hyperelliptic integrals. (d) We mention also Kötter’s solution [19,20] by quadratures in terms of hyperelliptic integrals of the integrable Clebsch’s [8] and Lyapunov–Steklov’s cases [35,46] of Kirchhoff’s equations describing the motion of a solid body in an ideal fluid, etc. The classical approach to proving that a system is integrable by quadratures (in terms of hyperelliptic integrals) was something very unsystematic and required a great deal of luck and ingenuity. Jacobi [16] himself was very much aware of this difficulty and in his famous “*Vorlesungen über Dynamik*”, in the context of geodesic flow on the ellipsoid (before introducing the elliptic coordinates), he wrote: “*Die Hauptschwierigkeit bei der Integration gegebener Differentialgleichungen scheint in der Einführung des richtigen Variablen zu bestehen, zu deren Auffindung es kein allgemeine Regel giebt. Man daher das umgekehrte Verfahren einschlagen und nach erlangter Kenntniss einer merkwürdigen Substitution die Probleme aufsuchen, bei welchen dieselbe mit Glück zu brauchen ist*” Finally, after Poincaré had recognized that most Hamiltonian systems are not completely integrable, the interest in this subject decreased for more than half a century.

2. Completely integrable systems and Kac–Moody Lie algebras

The discovery some 30 years ago (by Gardner et al. [12]) that the Korteweg–de Vries (KdV) equation [26]

$$\frac{\partial u}{\partial t} = \frac{1}{4} \left(6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right), \quad u(x, 0) = u(x), \quad x \in \mathbb{R} \tag{2.1}$$

could be integrated via inverse spectral methods has generated an enormous number of new ideas in the area of Hamiltonian completely integrable systems. Lax [24] showed that (2.1) is equivalent to the equation (called “Lax equation”):

$$\dot{A} = [B, A] \equiv BA - AB, \tag{2.2}$$

where A and B are the differential operators in x :

$$A = \frac{\partial^2}{\partial x^2} + u \text{ (Sturm–Liouville, Hill operator)}, \quad B = \frac{\partial^3}{\partial x^3} + \frac{3}{2}u \frac{\partial}{\partial x} + u.$$

Eq. (2.2) means that, under the time evolution of the system, the linear operator $A(t)$ remains similar to $A(0)$. So the spectrum of A is conserved, i.e. it undergoes an isospectral deformation. The eigenvalues of A , viewed as functionals, represent the integrals (constants of the motion) of the KdV equation. Around 1974, Mc Kean, van Moerbeke, Dubrovin and Novikov [9,36] solved the periodic problem for the KdV equation (for $x \in S^1$) in terms of a linear motion on a real torus. This torus is the real part of the Jacobi variety of a hyperelliptic curve with branch points defined by the simple periodic and anti-periodic spectrum of A . Also the motion is a straight line in the variables of the Abel–Jacobi map (1.1).

A parallel theory related to Jacobi matrices had its origin in the periodic Toda problem which I shall now explain. The Toda lattice equations (discretized version of the KdV equation) motion of n particles with exponential restoring forces are governed by the following Hamiltonian:

$$H = \frac{1}{2} \sum_{k=1}^n p_k^2 + \sum_{k=1}^n e^{q_k - q_{k+1}}.$$

The masses can be displayed on the line ($q_{n+1} = \infty$, non-periodic Toda system) or on the circle ($q_{n+1} = q_1$, periodic Toda system). The Hamiltonian equations can be written as follows:

$$\dot{q}_k = \frac{\partial H}{\partial p_k} = p_k, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k} = -e^{q_k - q_{k+1}} + e^{q_{k-1} - q_k}. \tag{2.3}$$

Consider the infinite Jacobi matrix (symmetric and tridiagonal)

From the first equation in (2.5) we have $\Sigma b_k = \text{constant}$, and we normalize by requiring that $\Sigma b_k = 0$. Then Eqs. (2.5) are equivalent to the Lax equation

$$\dot{A}(h) = [B(h), A(h)],$$

where $A(h)$ is given above and

$$B(h) = \begin{pmatrix} 0 & a_1 & \cdots & \cdots & -a_n h^{-1} \\ -a_1 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_{n-1} \\ a_n h & \cdots & \cdots & -a_{n-1} & 0 \end{pmatrix}.$$

We note that

$$\dot{A}_k(h) = [B(h), A^k(h)], \quad k \in \mathbb{N}$$

and $\text{tr } A^k = \text{constant}$, $1 \leq k \leq n$, i.e. $\text{tr } A^k$ are n integrals for (2.5). These integrals are independent, in involution and the Toda lattice is a completely integrable Hamiltonian system.

Krichever [23] generalized these ideas to differential operators of any order, inspired by special examples of Zaharov–Shabat [49]. Among which is the important Kadomtsev–Petviashvili (KP) equation

$$\frac{3}{4} \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \left[\frac{\partial u}{\partial t} - \frac{1}{4} \left(6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right) \right],$$

KdV equation (2.1) and the equation of a non-linear string (Boussinesq equation):

$$3 \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x} \left(6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right) = 0.$$

Also this theory was generalized to difference operators of any order by van Moerbeke and Mumford [41]. They worked out a systematic method which provides an algebraic map from the invariant manifolds defined by the intersection of the constants of the motion to the Jacobi variety

$$\text{Jac}(\mathcal{C}) = H^1(\mathcal{O}_{\mathcal{C}}) / H^1(\mathcal{C}, \mathbb{Z})$$

of an algebraic curve \mathcal{C} associated to Lax equation. More precisely, we define the spectral curve \mathcal{C} associated to Lax equation to be the normalization of the complete algebraic (hyperelliptic) curve whose affine equation is

$$\mathcal{C} : \det(A(h) - zI) = \left(h + \frac{1}{h} \right) - P(z) = 0,$$

where $P(z)$ is a polynomial of degree n in z . The motion of the system is linear in the variables of the Abel–Jacobi map (1.1).

The symplectic structure $\sum dp_i \wedge dq_i$ of the non-periodic Toda lattice in Flaschka’s variables (2.4) is given by [39]

$$\sum_{j=2}^n db_j \wedge \sum_{i=j}^n \frac{da_i}{a_i}$$

and is the symplectic structure of a Kostant–Kirillov–Souriau orbit formed by the coadjoint action of the subgroup $G_N \subset SL(n)$ of lower triangular matrices on the dual N^* of its Lie algebra N . Also, Adler [1] and Lebedev–Manin [25] developed a similar theory in the context of the symplectic structure of the Korteweg–de Vries equation. These investigations gave rise to the Adler–Kostant–Symes theorem [1,18,47]:

Theorem 1. *Let \mathcal{L} be a Lie algebra paired with itself via a non-degenerate, ad-invariant bilinear form $\langle \cdot, \cdot \rangle$, \mathcal{L} having a vector space decomposition $\mathcal{L} = \mathcal{K} + \mathcal{N}$ with \mathcal{K} and \mathcal{N} Lie subalgebras. Then, with respect to $\langle \cdot, \cdot \rangle$, we have the splitting $\mathcal{L}^* = \mathcal{K}^\perp + \mathcal{N}^\perp$ and $\mathcal{N}^* = \mathcal{K}^\perp$ paired with \mathcal{N} via an induced form $\langle\langle \cdot, \cdot \rangle\rangle$ inherits the coadjoint symplectic structure of Kostant and Kirillov; its Poisson bracket between functions H_1 and H_2 on \mathcal{N}^* reads*

$$\{H_1, H_2\}(a) = \langle\langle a, [\nabla_{\mathcal{N}^*} H_1, \nabla_{\mathcal{N}^*} H_2] \rangle\rangle, \quad a \in \mathcal{N}^*.$$

Let $V \subset \mathcal{N}^$ be an invariant manifold under the above co-adjoint action of \mathcal{N} on \mathcal{N}^* and let $\mathcal{A}(V)$ be the algebra of functions defined on a neighborhood of V , invariant under the coadjoint action of \mathcal{L} (which is distinct from the $\mathcal{N} - \mathcal{N}^*$ action). Then the functions H in $\mathcal{A}(V)$ lead to commuting Hamiltonian vector fields of the Lax isospectral form*

$$\dot{a} = [a, pr_{\mathcal{K}}(\nabla H)], \quad pr_{\mathcal{K}} \text{ projection onto } \mathcal{K}$$

This theorem produces Hamiltonian systems having many commuting integrals; some precise results are known for interesting classes of orbits in both the case of finite and infinite dimensional Lie algebras usually lead to non-compact ones. Any finite dimensional Lie algebra \mathcal{L} with bracket $[\cdot, \cdot]$ and killing form $\langle \cdot, \cdot \rangle$ leads to an infinite dimensional formal Laurent series extension

$$\mathcal{L} = \sum_{-\infty}^N A_i h^i : A_i \in \mathcal{L}, \quad N \in \mathbb{Z} \text{ free}$$

with bracket

$$\left[\sum A_i h^i, \sum B_j h^j \right] = \sum_{i,j} [A_i, B_j] h^{i+j}$$

and ad-invariant, symmetric forms

$$\left\langle \sum A_i h^i, \sum B_j h^j \right\rangle_k = \sum_{i+j=-k} \langle A_i, B_j \rangle$$

depending on $k \in \mathbb{Z}$. The forms $\langle \cdot, \cdot \rangle_k$ are non-degenerate if $\langle \cdot, \cdot \rangle$ is so. Let $\mathcal{L}_{p,q}$ ($p \leq q$) be the vector space of powers of h between p and q . A first interesting class of problems

is obtained by taking $\mathcal{L} = \mathcal{G}l(n, \mathbb{R})$ and by putting the form $\langle \cdot, \cdot \rangle_1$ on the Kac–Moody extension. Then we have the decomposition into Lie subalgebras

$$\mathcal{L} = \mathcal{L}_{0,\infty} + \mathcal{L}_{-\infty,-1} = \mathcal{K} + \mathcal{N}$$

with $\mathcal{K} = \mathcal{K}^\perp$, $\mathcal{N} = \mathcal{N}^\perp$ and $\mathcal{K} = \mathcal{N}^*$. Consider the invariant manifold V_m , $m \geq 1$ in $\mathcal{K} = \mathcal{N}^*$, defined as

$$V_m = \left\{ A = \sum_{i=1}^{m-1} A_i h^i + \alpha h^m, \quad \alpha = \text{diag}(\alpha_1, \dots, \alpha_n) \text{ fixed} \right\}$$

with $\text{diag}(A_{m-1}) = 0$.

Theorem 2. *The manifold V_m has a natural symplectic structure, the functions $H = \langle f(Ah^{-j}), h^k \rangle_1$ on V_m for good functions f lead to complete integrable commuting Hamiltonian systems of the form*

$$\dot{A} = [A, \text{pr}_{\mathcal{K}}(f'(Ah^{-j})h^{k-j})], \quad A = \sum_{i=0}^{m-1} A_i h^i + \alpha h$$

and their trajectories are straight line motions on the Jacobian of the curve \mathcal{C} of genus $(n-1)(nm-2)/2$ defined by $P(z, h) = \det(A - zI) = 0$. The coefficients of this polynomial provide the orbit invariants of V_m and an independent set of integrals of the motion. (Of particular interest are the flows where $j = m, k = m + 1$ which have the following form:

$$\dot{A} = [A, \text{ad}_{\beta} \text{ad}_{\alpha}^{-1} A_{m-1} + \beta h], \quad \beta_i = f'(\alpha_i)$$

the flow depends on f through the relation $\beta_i = f'(\alpha_i)$ only.)

Another class is obtained by choosing any semi-simple Lie algebra L . Then the Kac–Moody extension \mathcal{L} equipped with the form $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_0$ has the natural level decomposition

$$\mathcal{L} = \sum_{i \in \mathbb{Z}} L_i, \quad [L_i, L_j] \subset L_{i+j}, \quad [L_0, L_0] = 0, \quad L_i^* = L_{-i}.$$

Let $B^+ = \sum_{i \geq 0} L_i$ and $B^- = \sum_{i < 0} L_i$. Then the product Lie algebra $\mathcal{L} \times \mathcal{L}$ has the following bracket and pairing:

$$[(l_1, l_2), (l'_1, l'_2)] = ([l_1, l'_1], -[l_2, l'_2]), \quad \langle (l_1, l_2), (l'_1, l'_2) \rangle = \langle l_1, l'_1 \rangle - \langle l_2, l'_2 \rangle.$$

It admits the decomposition into $\mathcal{K} + \mathcal{N}$, with

$$\begin{aligned} \mathcal{K} &= \{(l, -l) : l \in \mathcal{L}\}, \quad \mathcal{K}^\perp = \{(l, l) : l \in \mathcal{L}\}, \\ \mathcal{N} &= \{(l_-, l_+) : l_- \in B^-, l_+ \in B^+, \text{pr}_0(l_-) = \text{pr}_0(l_+)\}, \\ \mathcal{N}^\perp &= \{(l_-, l_+) : l_- \in B^-, l_+ \in B^+, \text{pr}_0(l_+ + l_-) = 0\}, \end{aligned}$$

where pr_0 denotes projection onto L_0 . Then from Theorem 1, the orbits in $\mathcal{N}^* = \mathcal{K}$ possess a lot of commuting Hamiltonian vector fields of Lax form:

Theorem 3. *The N -invariant manifold*

$$V_{-j,k} = \sum_{-j \leq i \leq k} L_i \subseteq \mathcal{L} \simeq \mathcal{K}^\perp$$

has a natural symplectic structure and the functions $H(l_1, l_2) = f(l_1)$ on $V_{-j,k}$ lead to commuting vector fields of the Lax form

$$\dot{l} = [l, (pr^+ - \frac{1}{2}pr_0)\nabla H], \quad pr^+ \text{ projection onto } B^+$$

their trajectories are straight line motions on the Jacobian of a curve defined by the characteristic polynomial of elements in $V_{-j,k}$.

To summarize the methods explained above, let us assume a Hamiltonian system having an associated Lax equation

$$\dot{A} = [B(h), A(h)], \tag{2.6}$$

where

$$A = \sum_{-p}^q A_k(t)h^k, \quad B = \sum_{-p}^q B_k(t)h^k$$

are finite Laurent series in a variable h whose coefficients are matrices depending on a parameter. Some Hamiltonian flows on Kostant–Kirillov coadjoint orbits in subalgebras of infinite dimensional Lie algebras (Kac–Moody Lie algebras) yield large classes of extended Lax pairs (2.6). A general statement leading to such situations is given by the Adler–Kostant–Symes theorem. Using the van Moerbeke–Mumford linearization method, Adler and van Moerbeke [2,3] showed that the linearized flow could be realized on the Jacobian variety $Jac(\mathcal{C})$ (or some subabelian variety of it) of an algebraic curve (spectral curve) associated to (2.6). To be precise, a Hamiltonian flow of the type (2.6) preserves the spectrum of A and therefore its characteristic polynomial $P(z, h) = \det(A(h) - zI)$. The curve $\mathcal{C} : P(z, h) = 0$, of genus g , is time independent, i.e., its coefficients are integrals of the motion (2.6). We then construct an algebraic map from the complex invariant manifolds of these Hamiltonian systems to the Jacobi variety $Jac(\mathcal{C})$ of the curve \mathcal{C} . Therefore all the complex flows generated by the constants of the motion are straight line motions on these Jacobi varieties, i.e. the linearizing equations are given by

$$\sum_{i=1}^g \int_{s_i(0)}^{s_i(t)} \omega_k = c_k t, \quad 0 \leq k \leq g,$$

where $\omega_1, \dots, \omega_g$ span the g -dimensional space of holomorphic differentials on the curve \mathcal{C} of genus g . In a unifying approach, Griffiths [13] has found necessary and sufficient conditions on $B(h)$ for the Lax flow (2.6) to be linearizable on the Jacobi variety of its spectral curve, without reference to Kac–Moody Lie algebras.

Among the systems which fit into this scheme are the Euler’s, Lagrange’s and Kowalewski’s rigid body motion, Jacobi’s geodesic flow on ellipsoids, Neumann’s problem, Clebsch’s

case of Kirchoff’s equation, the Toda systems as explained above, the isospectral flows for period band matrices, Nahm’s equations which arise in the study of monopoles, etc. [3,6,15,17,42,43].

Next I shall explain how these methods can be used on some examples.

2.1. The Euler rigid body motion

It expresses the free motion of a rigid body around a fixed point and is governed by the equations

$$\dot{m} = m \wedge \lambda m, \quad m \in \mathbb{R}^3, \quad \lambda m = (\lambda_1 m_1, \lambda_2 m_2, \lambda_3 m_3) \in \mathbb{R}^3 \tag{2.7}$$

with (m_1, m_2, m_3) the angular momentum in body coordinates and $(\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1})$ the principal moments of inertia $(\lambda_1 < \lambda_2 < \lambda_3)$. The system has two invariants

$$H_1 = m_1^2 + m_2^2 + m_3^2 = c_1, \quad H_2 = \lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2 = c_2. \tag{2.8}$$

Observe that (2.7) can be written as a Hamiltonian system

$$\dot{x} = J \frac{\partial H}{\partial x}, \quad x = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \in so(3)$$

with Hamiltonian $H = \frac{1}{2} H_2$. The system evolves on the intersection of the sphere $H_1 = c_1$ and the ellipsoid $H_2 = c_2$. In \mathbb{R}^3 , this intersection will be isomorphic to two circles (with $c_2/\lambda_3 < c_1 < c_2/\lambda_1$). The system (2.7) has the following explicit form:

$$\dot{m}_1 = (\lambda_3 - \lambda_2)m_2m_3, \quad \dot{m}_2 = (\lambda_1 - \lambda_3)m_1m_3, \quad \dot{m}_3 = (\lambda_2 - \lambda_1)m_1m_2. \tag{2.9}$$

We have that

$$\frac{dm_1}{m_2m_3} = (\lambda_3 - \lambda_2) dt, \tag{2.10}$$

Solving (2.8) for m_2 and m_3 , one finds

$$m_2 = \pm \sqrt{\frac{c_1\lambda_3 - c_2 + (\lambda_1 - \lambda_3)m_1^2}{\lambda_3 - \lambda_2}}, \quad m_3 = \pm \sqrt{\frac{c_2 - c_1\lambda_2 + (\lambda_2 - \lambda_1)m_1^2}{\lambda_3 - \lambda_2}}$$

and substituting these expressions into (2.10) leads to an elliptic integral

$$\int_{m_1(0)}^{m_1(t)} \frac{dx}{\sqrt{(x^2 + a)(x^2 + b)}} = ct,$$

where $a = (c_1\lambda_3 - c_2)/(\lambda_1 - \lambda_3)$, $b = (c_2 - c_1\lambda_2)/(\lambda_2 - \lambda_1)$, $c = \sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_1)}$ and with respect to the elliptic curve

$$w^2 = (z^2 + a)(z^2 + b). \tag{2.11}$$

Therefore, the system (2.7) can be integrated by quadratures and their solutions can be expressed in terms of theta-functions, according to the classical Jacobi inversion problem. As is well known, if one identifies vectors in \mathbb{R}^3 with skew-symmetric matrices by the rule

$$x = (m_1, m_2, m_3), \quad X = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix}$$

then $x \wedge y \mapsto [X, Y] = XY - YX$. Using this isomorphism, we write (2.7) as

$$\dot{X} = [X, \lambda X], \quad X \in so(3), \quad \lambda X \in so(3), \tag{2.12}$$

which may be regarded as the Lie algebra version of (2.7). The solution to (2.12) has the form $X(t) = O(t)L(t)O^T(t)$, where $O(t)$ is a one parameter sub-group of $SO(3)$. So the Hamiltonian flow (2.12) preserves the spectrum of X and therefore its characteristic polynomial

$$\det(X - zI) = -z(z^2 + m_1^2 + m_2^2 + m_3^2).$$

Unfortunately, the spectrum of a 3×3 skew-symmetric matrix provides only one piece of information; the conservation of energy does not appear as part of the spectral information. Therefore one is let to considering another formulation. The basic observation, due to Manakov [37], is that if

$$\forall X \quad [X, \beta] + [\alpha, \lambda X] = 0 \Leftrightarrow \begin{cases} \lambda_1 = \frac{\beta_3 - \beta_2}{\alpha_3 - \alpha_2}, \\ \lambda_1 = \frac{\beta_1 - \beta_3}{\alpha_1 - \alpha_3}, \\ \lambda_1 = \frac{\beta_2 - \beta_1}{\alpha_2 - \alpha_1}, \end{cases}$$

$$\alpha = \text{diag}(\alpha_1, \alpha_2, \alpha_3), \quad \beta = \text{diag}(\beta_1, \beta_2, \beta_3),$$

Eq. (2.12) is equivalent to

$$(X + \alpha h) = [X + \alpha h, \lambda X + \beta h]$$

with a formal indeterminate h . The characteristic polynomial of the matrix $X + \alpha h$ is given by

$$\det(X + \alpha h - zI) = \prod_{i=1}^3 (\alpha_i h - z) + \left(\sum_{i=1}^3 \alpha_i m_i^2 \right) h - \left(\sum_{i=1}^3 m_i^2 \right) z$$

and except for some constants, its coefficients are generated by $\sum m_i^2$ and $\sum \alpha_i m_i^2$, which yield the energy. The spectrum of the matrix $X + \alpha h$ as a function of $h \in \mathbb{C}$ is time independent and is given by the zeroes of the characteristic polynomial $\det(X + \alpha h - zI) = 0$, which defines an elliptic curve

$$z^2 \prod_{i=1}^3 (\alpha_i w - 1) + \left(\sum_{i=1}^3 \alpha_i m_i^2 \right) w - \sum_{i=1}^3 m_i^2 = 0, \quad w = \frac{h}{z}$$

isomorphic to the original elliptic curve (2.11). The linearized Euler flow can be realized on the Jacobian of this elliptic curve, i.e. the curve itself.

2.2. Manakov’s geodesic flow on $SO(4)$

Consider the group $SO(4)$ and its Lie algebra $so(4)$ to itself, via the customary inner product

$$\langle X, Y \rangle = -\frac{1}{2}tr(X \cdot Y),$$

where

$$X = (X_{ij})_{1 \leq i, j \leq 4} = \sum_{i=1}^6 x_i e_i = \begin{pmatrix} 0 & -x_3 & x_2 & -x_4 \\ x_3 & 0 & -x_1 & -x_5 \\ -x_2 & x_1 & 0 & -x_6 \\ x_4 & x_5 & x_6 & 0 \end{pmatrix} \in so(4).$$

A left invariant metric on $SO(4)$ is defined by a non-singular symmetric linear map $\lambda: so(4) \rightarrow so(4), X \mapsto \lambda \cdot X$ and by the following inner product: given two vectors gX and gY in the tangent space $SO(4)$ at the point $g \in SO(4)$

$$\langle gX, gY \rangle = \langle x, \lambda^{-1} \cdot Y \rangle$$

regardless of g . Then geodesic motion with regard to this metric takes the form (Euler–Arnold equations):

$$\dot{X} = [X, \lambda \cdot X] \tag{2.13}$$

with $\lambda \cdot X = (\lambda_{ij} X_{ij})_{1 \leq i, j \leq 4} = \sum_{i=1}^6 \lambda_i x_i e_i$. This flow is Hamiltonian with regard to the usual Kostant–Kirillov symplectic structure induced on the orbit

$$O = \{Ad_g^*(X) = g^{-1}Xg : g \in SO(4)\},$$

formed by the coadjoint action $Ad_g^*(X)$ of the group $SO(4)$ on the dual Lie algebra $so(4)^* \approx so(4)$. Let $z_1, z_2 \in so(4)$ and consider $\xi_1 = [X, z_1], \xi_2 = [X, z_2]$ as two tangent vectors to the orbit at the point $X \in so(4)$. Then the symplectic structure is defined by

$$\omega(X)(\xi_1, \xi_2) = \langle X, [z_1, z_2] \rangle.$$

This orbit is four-dimensional and is defined by setting two trivial invariants H_1 and H_2 equal to generic constants c_1 and c_2 :

$$\begin{aligned} H_1 &= \sqrt{\det X} = x_1x_4 + x_2x_5 + x_3x_6 = c_1, \\ H_2 &= -\frac{1}{2}tr(X^2) = \sum_{i=1}^6 x_i^2 = c^2. \end{aligned} \tag{2.14}$$

Functions H defined on the orbit lead to Hamiltonian vector fields

$$\dot{X} = [X, \nabla H].$$

In particular,

$$H = \frac{1}{2} \langle X, \lambda \cdot X \rangle = \frac{1}{2} \sum_{i=1}^6 \lambda_i x_i^2 \tag{2.15}$$

induces geodesic motion (2.13). The constants of the motion are given by the two quadratic invariants H_1, H_2 (2.14) and the Hamiltonian H (2.15). Since the system is Hamiltonian on a four-dimensional symplectic manifold

$$\{H_1 = c_1\} \cap \{H_2 = c_2\},$$

to make it completely integrable, one needs one extra independent invariant. The first step towards the complete integrability of (2.13) was done by Manakov [37] who observed that under conditions

$$\begin{aligned} \lambda_1 &= \frac{\beta_2 - \beta_3}{\alpha_2 - \alpha_3}, & \lambda_4 &= \frac{\beta_1 - \beta_4}{\alpha_1 - \alpha_4}, \\ \lambda_2 &= \frac{\beta_1 - \beta_3}{\alpha_1 - \alpha_3}, & \lambda_5 &= \frac{\beta_2 - \beta_4}{\alpha_2 - \alpha_4}, & \alpha_i, \beta_i \text{ arbitrary and } \prod_{i < j} (\alpha_i - \beta_j) &\neq 0, \\ \lambda_3 &= \frac{\beta_1 - \beta_2}{\alpha_1 - \alpha_2}, & \lambda_6 &= \frac{\beta_3 - \beta_4}{\alpha_3 - \alpha_4} \end{aligned} \tag{2.16}$$

the Lax flow (2.13) can be transformed into the following Lax-type equation (with an indeterminate h):

$$\left. \begin{aligned} (X + \alpha h) &= [X + \alpha h, \lambda X + \beta h] \\ \alpha &= \text{diag}(\alpha_1, \dots, \alpha_4) \\ \beta &= \text{diag}(\beta_1, \dots, \beta_4) \end{aligned} \right\} \iff \begin{cases} \dot{X} = [X, \lambda \cdot X] \Leftrightarrow (2.13) \\ [X, \beta] + [\alpha, \lambda \cdot X] = 0 \Leftrightarrow (2.16) \\ [\alpha, \beta] = 0 \text{ trivially satisfied} \\ \text{for diagonal matrices} \end{cases} \tag{2.17}$$

Consider the Kac–Moody extension ($n = 4$)

$$\mathcal{L} = gl(n) = \left\{ \sum_{-\infty}^N A_i h^i : A_i \in gl(n), N \in \mathbb{Z} \text{ free} \right\}$$

of $gl(n)$ with the bracket

$$[A(h), B(h)] = \left[\sum A_i h^i, \sum B_j h^j \right] = \sum_k \left(\sum_{i+j=k} [A_i, B_j] \right) h^k$$

and the Killing form (ad-invariant form)

$$\langle A(h), B(h) \rangle = \left\langle \sum A_i h^i, \sum B_j h^j \right\rangle = \sum_{i+j=-1} \text{tr}(A_i B_j).$$

This Lie algebra has a natural decomposition

$$\mathcal{L} = \mathcal{K} + \mathcal{N}, \quad \mathcal{K} = \left\{ \sum_{i \geq 0} A_i h^i \right\}, \quad \mathcal{N} = \left\{ \sum_{i < 0} B_i h^i \right\}.$$

According to the Adler–Kostant–Symes theorem, the flow (2.17) is Hamiltonian on an orbit through the point $X + ah$, $X \in so(4)$, formed by the coadjoint action of $G_{\mathcal{N}}$ on the dual Kac–Moody algebra $\mathcal{N}^* = \mathcal{K}^\perp = \mathcal{K}$, where \perp is taken with respect the Killing form above. As a consequence, the coefficients of $z^i h^i$ appearing in the spectral curve

$$\mathcal{C} : \det(X + ah - zI) = 0$$

are invariant of the system in involution for the symplectic structure of this orbit. Also the flows generated by these invariants can be realized as straight lines on $Jac(\mathcal{C})$. More precisely, the spectral curve \mathcal{C} is given by

$$\mathcal{C} : \prod_{i=1}^4 (\alpha_i h - z) + H_2(X)z^2 - H_3(X)zh + H_4(X)h^2 + H_1^2(X) = 0 \tag{2.18}$$

with $H_1(X) = c_1$, $H_2(X) = c_2$ defined by (2.14), $H_3(X) = 2H = c_3$ by (2.15) and a 4th quadratic invariant of the form

$$H_4(X) = \sum_{i=1}^6 \mu_i x_i^2 = c_4,$$

where

$$\begin{aligned} \mu_1 &= \frac{\gamma_2 - \gamma_3}{\alpha_2 - \alpha_3}, & \mu_4 &= \frac{\gamma_1 - \gamma_4}{\alpha_1 - \alpha_4}, \\ \mu_2 &= \frac{\gamma_1 - \gamma_3}{\alpha_2 - \alpha_3}, & \mu_5 &= \frac{\gamma_2 - \gamma_4}{\alpha_2 - \alpha_4}, \\ \mu_3 &= \frac{\gamma_1 - \gamma_2}{\alpha_1 - \alpha_2}, & \mu_6 &= \frac{\gamma_3 - \gamma_4}{\alpha_3 - \alpha_4}. \end{aligned}$$

For generic choice of the c_i , \mathcal{C} is an algebraic curve of genus 3 and it has a natural involution

$$\tau : \mathcal{C} \rightarrow \mathcal{C}, \quad (z, h) \mapsto (-z, h)$$

due to the skew-symmetry of the matrix X . Therefore the Jacobian variety $Jac(\mathcal{C})$ of \mathcal{C} splits up into an even and odd part: the even part is an elliptic curve \mathcal{C}/τ and the odd part is a two-dimensional abelian surface $\text{Prym}_\tau(\mathcal{C})$ called the Prym variety of \mathcal{C}

$$Jac(\mathcal{C}) = \mathcal{C}/\tau + \text{Prym}_\tau(\mathcal{C}).$$

The van Moerbeke–Mumford linearization method provides then an algebraic map from the complex affine variety

$$A_c = \bigcap_{i=1}^4 \{H_i(X) = c_i\} \subset \mathbb{C}^6$$

to the Jacobi variety $Jac(\mathcal{C})$. By the anti-symmetry of \mathcal{C} , this map sends A_c to the Prym variety $\text{Prym}_\tau(\mathcal{C})$

$$A_c \rightarrow \text{Prym}_\tau(\mathcal{C}), \quad p \in A_c \mapsto \sum_{k=1}^3 s_k \in \text{Prym}_\tau(\mathcal{C})$$

and the complex flows generated by the constants of the motion are straight lines on $\text{Pr}ym_\tau(C)$.

3. Algebraic complete integrability

Deciding whether a system is completely integrable in the C^∞ -sense is a hopeless task at this stage. There are some techniques to prove the non-analytic integrability, i.e. the non-existence of analytic first integrals: in small dimensions, the Melnikov method [38] enables one to prove the non-integrability of a system very close to an integrable system by showing the existence of homoclinic points on the separatrices of the perturbed system. Another method consists in proving the absence of extra analytic integrals (beside the energy) for the linearized equations along some special a priori known solutions. These methods combined with other ideas, has been used by Ziglin [50–52] in the dynamics of the rigid body to show that the only analytically integrable cases among bodies with fixed points are the three known cases (Euler, Lagrange and Kowalewski) and Kozlov [22] for some classes of geodesic flows on the Euclidean group $E(3)$.

As mentioned above, the resolution of the KdV equation has led to unexpected connections between mechanics, spectral theory, Lie algebra theory, algebraic geometry and even differential geometry. All these connections have generated renewed interest in the questions around complete integrability of finite and infinite dimensional systems, ordinary and partial differential equations. However given a Hamiltonian system, it remains often hard to fit it into any of those general frameworks. But luckily, most of the problems under considerations possess the much richer structure of the so-called “*algebraic complete integrability*” which is more restrictive than the real analytic one commonly used. This notion will be motivated by the following example: consider again the Euler rigid body motion (Section 2.1). The system of differential equations (2.12) has Laurent series solutions,

$$X(t) = t^{-1}(X^{(0)} + X^{(1)}t + X^{(2)}t^2 + \dots), \tag{3.1}$$

depending on $\dim(\text{phase space}) - 1 = 2$ free parameters. Putting (3.1) into (2.12), solving inductively for the $X^{(k)}$, one finds at the 0th step a non-linear equation,

$$X^{(0)} + [X^{(0)}, \lambda X^{(0)}] = 0 \tag{3.2}$$

and at the k th step, a linear system of equations,

$$(L - kI)X^{(k)} = - \sum_{i=1}^{k-1} [X^{(i)}, \lambda X^{(k-i)}], \quad k \geq 1,$$

where L is the linear operator $L: so(3) \rightarrow so(3)$ defined by

$$L(Y) = Y + [X^{(0)}, \lambda Y] + [Y, \lambda X^{(0)}] = \text{Jacobian of (3.2)}$$

and the matrix $X^{(0)}$ appearing in L is a solution of the non-linear equation (3.2). An easy computation shows that the matrix $(L - kI)$ is always invertible unless $k = 2$ and then

the rank equals 1. This shows that the coefficient $X^{(2)}$ contains two free parameters, which account for c_1 and c_2 . The complex intersection

$$\bigcap_{i=1}^2 \{H_i = c_i\} \subseteq \mathbb{C}^3$$

is the affine part of an elliptic curve $\subseteq \mathbb{C}\mathbb{P}^3$ which is obviously a torus and it is well known that this torus has an algebraic addition law connecting $p(t_1 + t_2)$ to $p(t_1)$ and $p(t_2)$ where $p(t) \equiv (m_1(t), m_2(t), m_3(t))$ is a solution of (2.7). This state of affairs is summarized by the notion of *algebraic complete integrability*, it was introduced by Adler–van Moerbeke [4,5] and Mumford [44].

Consider a Hamiltonian system

$$\dot{z} = f(z) = J \frac{\partial H}{\partial z}, \quad z \in \mathbb{R}^m, \tag{3.3}$$

where $J = J(z)$ is a skew-symmetric matrix with polynomial entries in z , for which the corresponding Poisson bracket

$$\{H_i, H_j\} = \left\langle \frac{\partial H_i}{\partial z}, J \frac{\partial H_j}{\partial z} \right\rangle$$

satisfies the Jacobi identity. The system (3.3) with polynomial right-hand side will be called *algebraically completely integrable (a.c.i.)* when:

(a) The system possesses $n + k$ polynomial invariants H_1, \dots, H_{n+k} (Casimir functions) of which k lead to zero vector fields $J(\partial H_{n+i}/\partial z) = 0, 1 \leq i \leq k$, the n remaining ones are in involution (i.e. $\{H_i, H_j\} = 0$) and $m = 2n + k$. For most values of $c_i \in \mathbb{R}$, the invariant manifolds $\bigcap_{i=1}^{n+k} \{H_i = c_i, z \in \mathbb{R}^m\}$ are compact and connected. Then, according to the Arnold–Liouville theorem, there exists a diffeomorphism

$$\Phi : \bigcap_{i=1}^{n+k} \{H_i = c_i, z \in \mathbb{R}^m\} \rightarrow \mathbb{R}^n / \text{Lattice} \quad (\text{real tori})$$

and the solutions of the system (3.3) are straight line motions on these tori.

(b) The invariant manifolds, thought of as affine varieties in \mathbb{C}^m (non-compact) can be completed into complex algebraic tori, i.e.,

$$\bigcap_{i=1}^{n+k} \{H_i = c_i, z \in \mathbb{C}^m\} = \text{complex algebraic torus } \mathbb{C}^n / \text{Lattice} \text{ (i.e. abelian variety) } \setminus \text{a divisor } D \text{ (i.e. one or several codimension 1 subvarieties).}$$

Algebraic means that $\mathbb{C}^n / \text{Lattice}$ can be defined as an intersection $\bigcap_{i=1}^N \{F_i(Z_0, \dots, Z_N) = 0\}$ involving a large number of homogeneous polynomials F_i . The functions z_i are required to be meromorphic on $\mathbb{C}^n / \text{Lattice}$, and in particular in the natural coordinates (t_1, \dots, t_n) of $\mathbb{C}^n / \text{Lattice}$ coming from \mathbb{C}^n , the functions $z_i = z_i(t_1, \dots, t_n)$ are meromorphic and (3.3)

defines straight line motion on $\mathbb{C}^n / \text{Lattice}$. Condition (b) means, in particular, there is an algebraic map

$$(z_1(t), \dots, z_m(t)) \mapsto (s_1(t), \dots, s_n(t))$$

making the following sums linear in t :

$$\sum_{i=1}^n \int_{s_i(0)}^{s_i(t)} \omega_j = a_j t, \quad 1 \leq j \leq n,$$

where $\omega_1, \dots, \omega_n$ denote holomorphic differentials on some algebraic curves. In particular, when these curves are hyperelliptic, the above sums coincide with the formula (1.1).

Adler and van Moerbeke [5] have developed and used the following necessary algebraic complete integrability criterion:

Theorem 4. *If the Hamiltonian system (3.3) is algebraically completely integrable, then*

- (i) *each z_i blows up for some value of $t \in \mathbb{C}$.*
- (ii) *whenever it blows up, the solution $z(t)$ behaves as a Laurent series*

$$z_i = t^{-k_i} (z_i^{(0)} + z_i^{(1)} t + z_i^{(2)} t^2 + \dots), \quad k_i \in \mathbb{Z}, \quad \text{some } k_i > 0, \tag{3.4}$$

which admits $m - 1$ free parameters.

This a.c.i. criterion is implicit in a beautiful investigation of Kowalewski [21] in 1889 for which she was awarded the Bordin prize by the French Academy; there she finds all the completely integrable rigid body motion: the Euler rigid body motion, the Lagrange top and her famous Kowalewski top. To explain the criterion, if the Hamiltonian flow (3.3) is algebraically completely integrable, it means that the variables z_i are meromorphic on the torus $\mathbb{C}^n / \text{Lattice}$ and by compactness they must blow up along a codimension 1 subvariety (a divisor) $D \subset \mathbb{C}^n / \text{Lattice}$. By the a.c.i. definition, the flow (3.3) is a straight line motion in $\mathbb{C}^n / \text{Lattice}$ and thus it must hit the divisor D in at least one place. Moreover, through every point of D , there is a straight line motion and therefore a Laurent expansion around that point of intersection. Hence the differential equation must admit Laurent expansions which depend on the $n - 1$ parameters defining D and the $n + k$ constants c_i defining the torus $\mathbb{C}^n / \text{Lattice}$, the total count is therefore $m - 1 = \dim(\text{phase space}) - 1$ parameters.

Assume now Hamiltonian flows to be (weight)-homogeneous with a weight $\nu_i \in \mathbb{Z}, > 0$, going with each variable z_i , i.e.

$$f_i(\alpha^{\nu_1} z_1, \dots, \alpha^{\nu_m} z_m) = \alpha^{\nu_i+1} f_i(z_1, \dots, z_m), \quad \forall \alpha \in \mathbb{C}.$$

Observe that then the constants of the motion H can be chosen to be (weight)-homogeneous:

$$H(\alpha^{\nu_1} z_1, \dots, \alpha^{\nu_m} z_m) = \alpha^k H(z_1, \dots, z_m), \quad k \in \mathbb{Z}.$$

According to Yosida [48], if the flow is algebraically completely integrable, the differential equations (3.3) must admit Laurent series solutions (3.4) depending on $m - 1$ free parameters.

We must have $k_i = v_i$ and coefficients in the series must satisfy at the 0th step non-linear equations,

$$f_i(z_1^{(0)}, \dots, z_m^{(0)}) + g_i z_i^{(0)} = 0, \quad 1 \leq i \leq m \tag{3.5}$$

and at the k th step, linear systems of equations

$$(\mathbf{L} - k\mathbf{I})z^{(k)} = \begin{cases} 0 & \text{for } k = 1, \\ \text{some polynomial in } z^{(1)}, \dots, z^{(k-1)} & \text{for } k > 1, \end{cases} \tag{3.6}$$

where

$$\mathbf{L} = \text{Jacobian map of (3.5)} = \frac{\partial f}{\partial z} + g\mathbf{I}|_{z=z^{(0)}}.$$

If $m - 1$ free parameters are to appear in the Laurent series, they must either come from the non-linear equations (3.5) or from the eigenvalue problem (3.6), i.e. \mathbf{L} must have at least $m - 1$ integer eigenvalues. These are much less conditions than expected, because of the fact that the homogeneity k of the constant H must be an eigenvalue of \mathbf{L} . Moreover, the formal series solutions are automatically convergent as a consequence of the majorant method [31].

To show that a system is algebraically completely integrable, we proceed as follows:

- The first step is to show the existence of the Laurent solutions, which requires an argument precisely every time k is an integer eigenvalue of \mathbf{L} and therefore $\mathbf{L} - k\mathbf{I}$ is not invertible.
- One shows the existence of the remaining constants of the motion in involution so as to reach the number $n + k$.
- For given c_1, \dots, c_m , the set

$$D \equiv \left\{ \begin{array}{l} \text{Laurent solutions } z_i(t) = t^{-v_i} (z_i^{(0)} + z_i^{(1)}t + z_i^{(2)}t^2 + \dots), \\ 1 \leq i \leq m, \text{ such that } H_j(z_i(t)) = c_j + \text{Taylor part} \end{array} \right\}$$

defines one or several $n - 1$ dimensional algebraic varieties (divisor) having the property that

$$\bigcap_{i=1}^{n+k} \{H = c_i, z \in \mathbb{C}^m\} \cup D = \text{a smooth compact, connected variety with } n \text{ commuting vector fields independent at every point} \\ = \text{a complex algebraic torus } T^n = \mathbb{C}^n / \text{Lattice}.$$

The flows $J(\partial H_{k+i} / \partial z), \dots, J(\partial H_{k+n} / \partial z)$ are straight line motions on T^n .

- From the divisor D , a lot of information can be obtained with regard to the periods and the action–angle variables.

Next I shall explain how these ideas can be used on an interesting completely integrable Hamiltonian system.

3.1. Kowalewski’s top

The motion for Kowalewski’s top is governed by the equations

$$\dot{m} = m \wedge \lambda m + \gamma \wedge l, \quad \dot{\gamma} = \gamma \wedge \lambda m, \tag{3.7}$$

where m , γ and l denote, respectively, the angular momentum, the directional cosine of the z -axis (fixed in space), the center of gravity which after some rescaling and normalization may be taken as $l = (1, 0, 0)$ and $\lambda m = (m_1/2, m_2/2, m_3/2)$. The system (3.7) can be written

$$\begin{aligned} \dot{m}_1 &= m_2 m_3, & \dot{\gamma}_1 &= 2m_3 \gamma_2 - m_2 \gamma_3, \\ \dot{m}_2 &= -m_1 m_3 + 2\gamma_3, & \dot{\gamma}_2 &= m_1 \gamma_3 - 2m_3 \gamma_1, \\ \dot{m}_3 &= -2\gamma_2, & \dot{\gamma}_3 &= m_2 \gamma_1 - m_1 \gamma_2, \end{aligned} \tag{3.8}$$

with constants of motion

$$\begin{aligned} H_1 &= \frac{1}{2}(m_1^2 + m_2^2) + m_3^2 + 2\gamma_1 = c_1, \\ H_2 &= m_1 \gamma_1 + m_2 \gamma_2 + m_3 \gamma_3 = c_2, \\ H_3 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = c_3 = 1, \\ H_4 &= \left[\left(\frac{m_1 + im_2}{2} \right)^2 - (\gamma_1 + i\gamma_2) \right] \left[\left(\frac{m_1 - im_2}{2} \right)^2 - (\gamma_1 - i\gamma_2) \right] = c_4. \end{aligned} \tag{3.9}$$

The system (3.8) admits Laurent series solutions

$$\begin{aligned} m_i(t) &= t^{-1}(m_i^{(0)} + m_i^{(1)}t + m_i^{(2)}t^2 + \dots), \\ \gamma_i(t) &= t^{-2}(\gamma_i^{(0)} + \gamma_i^{(1)}t + \gamma_i^{(2)}t^2 + \dots), \end{aligned} \tag{3.10}$$

which depend on five free parameters $\alpha_1, \dots, \alpha_5$. Putting (3.10) into the differential equations (3.8), one finds at the 0th step a non-linear system

$$\begin{aligned} m_1^{(0)} + m_2^{(0)} m_3^{(0)} &= 0, & 2\gamma_1^{(0)} + 2m_3^{(0)} \gamma_2^{(0)} - m_2^{(0)} \gamma_3^{(0)} &= 0, \\ m_2^{(0)} - m_1^{(0)} m_3^{(0)} + 2\gamma_3^{(0)} &= 0, & 2\gamma_2^{(0)} + m_1^{(0)} \gamma_3^{(0)} - 2m_3^{(0)} \gamma_1^{(0)} &= 0, \\ m_3^{(0)} - 2\gamma_2^{(0)} &= 0, & 2\gamma_3^{(0)} + m_2^{(0)} \gamma_1^{(0)} - m_1^{(0)} \gamma_2^{(0)} &= 0, \end{aligned} \tag{3.11}$$

and at the k th step, a system of linear equations

$$(\mathbf{L} - k\mathbf{I}) \begin{pmatrix} m^{(k)} \\ \gamma^{(k)} \end{pmatrix} = \begin{cases} 0 & \text{for } k = 1, \\ \text{a polynomial in } \left(\begin{pmatrix} m^{(1)} \\ \gamma^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} m^{(k-1)} \\ \gamma^{(k-1)} \end{pmatrix} \right), & k \geq 1, \end{cases}$$

where L is the Jacobian matrix of the equations (3.11). The parameter α_1 appears at the 0th step, i.e. in the resolution of (3.11) and the four remaining ones $\alpha_2, \dots, \alpha_5$ at the k th step, $k = 1, \dots, 4$. There are two distinct families of Laurent solutions,

First Laurent solutions

Second Laurent solutions

$$m_1(t) = \frac{\alpha_1}{t} + i(\alpha_1^2 - 2)\alpha_2 + o(t)$$

$$m_1(t) = \frac{\alpha_1}{t} - i(\alpha_1^2 - 2)\alpha_2 + o(t)$$

$$m_2(t) = \frac{i\alpha_1}{t} - \alpha_1^2\alpha_2 + o(t)$$

$$m_2(t) = \frac{-i\alpha_1}{t} - \alpha_1^2\alpha_2 + o(t)$$

$$m_3(t) = \frac{i}{t} + \alpha_1\alpha_2 + o(t)$$

$$m_3(t) = \frac{-i}{t} + \alpha_1\alpha_2 + o(t)$$

$$\gamma_1(t) = \frac{1}{2t^2} + o(t)$$

$$\gamma_1(t) = \frac{1}{2t^2} + o(t)$$

$$\gamma_2(t) = \frac{i}{2t^2} + o(t)$$

$$\gamma_2(t) = \frac{-i}{2t^2} + o(t)$$

$$\gamma_3(t) = \frac{\alpha_2}{t} + o(t)$$

$$\gamma_3(t) = \frac{\alpha_2}{t} + o(t)$$

By substituting these series in the constants of the motion H_i (3.9), one eliminates three parameters linearly, leading to algebraic relation between the two remaining parameters, which is nothing but the equation of the divisor D along which the m_i, γ_i blow up. Since the system (3.8) admits two families of Laurent solutions, then D is a set of two isomorphic curves of genus 3, $D = D_1 + D_{-1}$:

$$D_\epsilon : P(\alpha_1, \alpha_2) = (\alpha_1^2 - 1)((\alpha_1^2 - 1)\alpha_2^2 - P(\alpha_2)) + c_4 = 0, \tag{3.12}$$

where $P(\alpha_2) = c_1\alpha_2^2 - 2\epsilon c_2\alpha_2 - 1$ and $\epsilon = \pm 1$. Each of the curve D_ϵ is a 2 – 1 ramified cover $(\alpha_1, \alpha_2, \beta)$ of elliptic curves D_ϵ^0 :

$$D_\epsilon^0 : \beta^2 = P^2(\alpha_2) - 4c_4\alpha_2^4 \tag{3.13}$$

ramified at the four points $\alpha_1 = 0$ covering the four roots of $P(\alpha_2) = 0$. It was shown by the author [27–29] that each divisor D_ϵ is ample and defines a polarization (1,2), whereas the divisor D , of geometric genus 9, is very ample and defines a polarization (2,4). More precisely, we have the following theorem:

Theorem 5. *Let*

$$A_c = \bigcap_{i=1}^4 \{H_i = c_i\} \subset \mathbb{C}^6$$

be the affine surface defined by putting the four invariants (3.9) of the Kowalewski flow (3.8) equal to generic constants, then

(a) A_c is the affine part of an abelian surface \tilde{A}_c with

$\tilde{A}_c/A_c = D =$ one genus 9 curve consisting of two genus 3 curves D_ϵ (3.12) intersecting in four points. Each D_ϵ is a double cover of an elliptic curve D_ϵ^0 (3.13) ramified at four points.

Moreover, the Hamiltonian flows generated by the vector fields X_{H_1} and X_{H_4} are straight lines on \tilde{A}_c .

(b) The eight functions

$$1, f_1 = m_1, f_2 = m_2, f_3 = m_3, f_4 = \gamma_3, f_5 = f_1^2 + f_2^2, \\ f_6 = 4f_1 f_4 - f_3 f_5, f_7 = (f_2 \gamma_1 - f_1 \gamma_2) f_3 + 2f_4 \gamma_2.$$

form a basis of the vector space $\mathcal{L}(D)$ of meromorphic functions on \tilde{A}_c with at worst a simple pole along D . Moreover, the map

$$\tilde{A}_c \simeq \mathbb{C}^2 / \text{Lattice} \rightarrow \mathbb{C}\mathbb{P}^7, \quad (t_1, t_2) \mapsto [(1, f_1(t_1, t_2), \dots, f_7(t_1, t_2))]$$

is an embedding of \tilde{A}_c into $\mathbb{C}\mathbb{P}^7$.

(c) The tori can be identified as

$$\tilde{A}_c = \text{dual of } \text{Prym}(D_\epsilon / D_\epsilon^0).$$

The method explained above can be and has been applied to other problems. For example, we show in the same style as before (see [14,33]), that the Manakov geodesic flow on $SO(4)$ is algebraically completely integrable and the abelian surface A_c which completes the affine surface defined by the four constants is the Prym variety $\text{Prym}_\tau(D)$ of genus 3 curve D :

$$D : \begin{cases} w^2 + c_2 z^2 - c_3 z + c_4 + 2c_1 y = 0, \\ y^2 = \prod_{i=1}^4 (z - a_i). \end{cases}$$

We know from Section 3.1, that this problem linearizes on the Prym variety $\text{Prym}_\tau(C)$ of the spectral curve C (2.18). $\text{Prym}_\tau(D)$ is not isomorphic to $\text{Prym}_\tau(C)$ but only isogenous, the precise relation between these two abelian surfaces begin that they are dual of each other; in fact, the functions x_i are themselves meromorphic on $\text{Prym}_\tau(D)$, while only their squares are on $\text{Prym}_\tau(C)$. It is also isogeneous to the Jacobian of a naturally arising hyperelliptic curve, as follows from Kötter’s [19,20] investigation of the Clebsch case for the motion of a rigid body in an ideal fluid. The connection with Kowalewski’s top and Hénon–Heiles motion can be found in Adler–van Moerbeke [6].

References

[1] M. Adler, On a trace functional for pseudo-differential operators and the symplectic structure of the Korteweg–de Vries equation, *Invent. Math.* 50 (1979) 219–248.
 [2] M. Adler, P. van Moerbeke, Completely integrable systems, Euclidean Lie algebras and curves, *Adv. Math.* 38 (1980) 267–317.
 [3] M. Adler, P. van Moerbeke, Linearization of Hamiltonian systems, Jacobi varieties and representation theory, *Adv. Math.* 38 (1980) 318–379.
 [4] M. Adler, P. van Moerbeke, Kowalewski’s asymptotic method, Kac–Moody Lie algebras and regularization, *Comm. Math. Phys.* 83 (1982) 83–106.
 [5] M. Adler, P. van Moerbeke, The algebraic complete integrability of geodesic flow on $SO(4)$, *Invent. Math.* 67 (1982) 297–331.

- [6] M. Adler, P. van Moerbeke, The Kowalewski and Hénon–Heiles motions as Manakov geodesic flows on $SO(4)$ – A two-dimensional family of Lax pairs, *Comm. Math. Phys.* 113 (1988) 659–700.
- [7] V.I. Arnold, *Mathematical Methods in Classical Mechanics*, Springer, Berlin, 1978.
- [8] A. Clebsch, Der Bewegung eines starren Körpers in einer Flüssigkeit, *Math. Ann.* 3 (1971) 238–268.
- [9] B.A. Dubrovin, S.P. Novikov, Periodic and conditionally periodic analogues of multi-soliton solutions of the KdV equation, *Dokl. Akad. Nauk URSS* 6 (1974) 2131–2144.
- [10] H. Flaschka, The Toda lattice I, *Phys. Rev. B* 9 (1974) 1924–1925.
- [11] H. Flaschka, The Toda lattice II, *Progr. Theoret. Phys.* 51 (1974) 703–716.
- [12] C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, Method for solving the Korteweg–de Vries equation, *Phys. Rev. Lett.* 10 (1967) 1095–1097.
- [13] P.A. Griffiths, Linearizing flows and a comological interpretation of Lax equations, *Amer. J. Math.* 107 (1985) 1445–1483.
- [14] L. Haine, Geodesic flow on $SO(4)$ and Abelian surfaces, *Math. Ann.* 263 (1983) 435–472.
- [15] N.J. Hitchin, On the construction of monopoles, *Comm. Math. Phys.* 89 (1983) 145–190.
- [16] C.G.J. Jacobi, *Vorlesungen über Dynamik*, in: *Gesammelte Werke, Supplementband*, Berlin, 1980.
- [17] H. Knörrer, Geodesics on the ellipsoid, *Invent. Math.* 59 (1980) 119–144.
- [18] B. Kostant, The solution to a generalized Toda lattice and representation theory, *Adv. Math.* 34 (1979) 195–338.
- [19] F. Kötter, Über die Bewegung eines festen Körpers in einer Flüssigkeit I, II, *J. Rienc. Angew. Math.* 109 (1892) 51–81, 89–110.
- [20] F. Kötter, Die von Steklow und Lyapunov entdeckten integralen Fälle der Bewegung eines Körpers in einer Flüssigkeit Sitzungsber. Königlich Preussische Akad. d. Wiss., Berlin, vol. 6, 1900, pp. 79–87.
- [21] S. Kowalewski, Sur le problème de la rotation d'un corps solide autour d'un point fixe, *Acta Math.* 12 (1889) 177–232.
- [22] V.V. Kozlov, Integrability and non-integrability in Hamiltonian mechanics, *Uspekhi Mat. Nauk* 38 (1983) 3–67 (Transl 1, *Russian Math. Surveys* 38 (1983) 1–76).
- [23] M. Krichever, Algebraic-geometrical construction of the Zakharov Shabat equations and their periodic solutions, *Sov. Math. Dokl.* 17 (1976) 394–397.
- [24] P. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* 21 (1968) 467–490.
- [25] D.R. Lebedev, Yu.I. Manin, The Gelfand–Dikii Hamiltonian operator and the coadjoint representation of the Volterra group, *Func. Anal. Appl.* 13 (1979) 40–46.
- [26] A. Lesfari, Equation de Korteweg–de Vries, *Mem. Dpt. Maths. Fac. Sc., U.C.L., Louvain-la-Neuve, Belgique*, 1982, pp. 1–113.
- [27] A. Lesfari, Une approche systématique à la résolution du corps solide de Kowalewski, *C. R. Acad. Sc. Paris*, t. 302, Série I, 1986, pp. 347–350.
- [28] A. Lesfari, Une approche systématique à la résolution des systèmes intégrables. Proc. de la 2^e école de géométrie- analyse (15–19 juin 1987), p. 83, E.H.T.P., Casablanca, Maroc (Conférences organisées en l'honneur du Pr. A. Lichnerowicz).
- [29] A. Lesfari, Abelian surfaces and Kowalewski's top. *Ann. Scient. Ec. Norm. Sup.*, Paris, 4^e série, t. 21, 1988, pp. 193–223.
- [30] A. Lesfari, Systèmes Hamiltoniens algébriquement complètement intégrables, Preprint, Dpt. Maths. Fac. Sc., El-Jadida, Maroc, 1990, pp. 1–38.
- [31] A. Lesfari, *Eléments d'analyse*, Sochepress-Université, Casablanca, Maroc, 1991.
- [32] A. Lesfari, On affine surface that can be completed by a smooth curve, *Results Math.* 35 (1999) 107–118.
- [33] A. Lesfari, Geodesic flow on $SO(4)$, Kac–Moody Lie algebra and its singularities in the complex t -plane, *Publicacions Matemàtiques* 43 (1999), in press.
- [34] A. Lesfari, Une méthode de compactification de variétés liées aux systèmes dynamiques, *Cahiers de la recherche, Rectorat-Université Hassan II Ain Chock* 1(1) (1999) 147–157.
- [35] A.M. Lyapunov, Report of Kharkov. *Math. Soc. Ser. 2*, 4, n^{os} 1–2, 81–85 (1893) (*Gesammelte Werke*, vol. 1, 320–324).
- [36] H.P. Mc Kean, P. van Moerbeke, The spectrum of Hill's equation, *Invent. Math.* 30 (1975) 217–274.
- [37] S.V. Manakov, Remarks on the integrals of the Euler equations of the n -dimensional heavy top, *Func. Anal. Appl.* 10 (1976) 93–94.
- [38] V.K. Melnikov, On the stability of the center for time-periodic perturbations, *Trans. Mosc. Math. Soc.* 12 (1963) 1–57.

- [39] P. van Moerbeke, The spectrum of Jacobi matrices, *Invent. Math.* 37 (1976) 45–81.
- [40] P. van Moerbeke, Algebraic complete integrability of Hamiltonian systems and Kac–Moody Lie algebras, *Proc. Int. Cong. Math.*, Warszawa, 1983.
- [41] P. van Moerbeke, D. Mumford, The spectrum of difference operators and algebraic curves, *Acta Math.* 143 (1979) 93–154.
- [42] J. Moser, Three integrable Hamiltonian systems connected with isospectral deformations, *Adv. Math.* 16 (1975) 197–219.
- [43] J. Moser, Geometry of quadrics and spectral theory, Lecture delivered at the symposium in honor of S.S. Chern, Berkeley, 1979, Springer, Berlin, 1980.
- [44] D. Mumford, Tata lectures on theta. II. *Progress in Math.*, Birkhäuser, Boston, 1982.
- [45] C. Neumann, De problemate quodam mechanica, quod ad primam integralium ultraellipticorum classem revocatur, *Reine und Angew. Math. J.* 56 (1859) 44–63.
- [46] V. Steklov, On the motion of a solid in a liquid, *Math. Ann.* 42 (1893) 273–394.
- [47] W. Symes, Systems of Toda type, Inverse spectral problems and representation theory, *Invent. Math.* 59 (1980) 13–53.
- [48] H. Yoshida, Necessary conditions for the existence of algebraic first integrals. I. Kowalewski’s exponents. II. Conditions for algebraic integrability. *Celestial Mech.* 31 (1983) 363–379, 381–399.
- [49] V. Zaharov, A. Shabat, A schema for integrating the non-linear equations of Math. Physics by the method of the inverse scattering problem I. *Funct. Anal. Appl.* 8 (1974).
- [50] S.L. Ziglin, Splitting of separatrices, branching of solutions and non-existence of an integral in the dynamics of a solid body. *Trans. Moscow Math. Soc.* 41 (1980) 287–303 (Transl. 1 (1982) 283–298).
- [51] S.L. Ziglin, Branching of solutions and non-existence of first integrals in Hamiltonian mechanics. 1. *Funct. Anal. Appl.* 16 (1982) 30–41.
- [52] S.L. Ziglin, Branching of solutions and non-existence of first integrals in Hamiltonian mechanics. 2. *Funct. Anal. Appl.* 17 (1983) 8–23.